DERIVATION OF CONVECTIVE KIRCHHOFF’S FORMULA FOR FAR-FIELD PROPAGATION OF SOUND

Yong Woo Lee*               Duck Joo Lee**
Doctoral Candidate            Professor
Division of Aerospace Engineering
School of Mechanical Aerospace & Systems Engineering
KAIST
Daejeon, Republic of Korea
(Tele: +82-42-350-3756    Fax: +82-42-350-3710)
(e-mail: lyw0728@kaist.ac.kr, djlee@kaist.ac.kr)

ABSTRACT

The Kirchhoff’s formula is a useful formula for the wave propagation. The wave property at a desired observer position can be obtained through a surface integration for closed surface surrounding the source, so-called the Kirchhoff’s surface. The classical Kirchhoff’s formula is based on the wave equation for stationary medium. This paper describes a Kirchhoff’s formula for the convective wave equation, which includes a moving medium effect. The generalized function theory is used for derivation of generalized convective wave equation. The integral form solution, the Kirchhoff’s formula, is obtained using a convective Green’s function for an arbitrarily moving surface. The derived formula has similar form with the classical Kirchhoff’s formula but an additional term appears due to a moving medium effect. For convenience, manipulation is applied to the additional term and a final form like the classical formula is obtained.

INTRODUCTION

The Kirchhoff’s formula is one of useful formula for a sound radiation problem. The wave property at a desired observer position can be obtained through a designated surface integration. The original Kirchhoff’s formula is derived for a stationary surface. But Farassat et al.[1] extended the formula for a moving surface for predicting the noise of propellers and rotors. Farassat et al. used the generalized function theory to derive a wave equation valid for whole region with discontinuity.

The classical and extended Kirchhoff’s formula is based on the non-convective wave equation. Thus, a Galilean transformation is required to predict the noise in moving medium. But if the medium moves with constant speed, a new formula containing the moving medium effect can be more convenient and efficient. Blokhintzev[2] derived a Kirchhoff’s formula for convective wave equation using coordinate transformation. So the integration has to be conducted in transformed coordinate and it caused inconvenient. Wu et al.[3] derived a integral formula in the physical coordinates to use the integral formula without coordinate transformation. But the Wu et al.’s formula is based on the Helmholtz equation. Accordingly the formula can be used for a harmonic wave only.

et al.\cite{7} derived convective acoustic analogy formula using a similar procedure of Farassat et al.’s. The acoustic analogy requires all aerodynamic variables and this can be obtained from CFD.

The Kirchhoff’s formula is linear and it may look like lower level formula than acoustic analogy. But Kirchhoff’s formula requires only one variable and its temporal and spatial derivatives. Thus, it can be more useful for some problems. Furthermore, the Kirchhoff’s formula can be used as the equation if we set the observer position on the surface. By using this integral equation, a scattering problem can be solved. And the convective Kirchhoff’s formula can be used for an aerodynamic problem also. The convective wave equation is a linearized perturbed potential equation and if the potential is steady, the equation will be changed to well-known linearized small perturbation potential equation. So, for thin propeller blades or rotor blades, aerodynamics and aeroacoustics can be solved simultaneously.

In this paper, we derived the Kirchhoff’s formula for the convective wave equation in the physical coordinates. The Kirchhoff’s surface is assumed as moving surface. The final form is arranged to the similar form of the classical Kirchhoff’s formula except an additional term due to the moving medium effect. The additional term is modified to have the same form with the classical Kirchhoff’s formula for easy numerical implementation.

**THE GENERALIZED CONVETIVE WAVE EQUATION**

The convective wave equation is changed to the generalized convective wave equation using the generalized function theory. The original convective wave equation is valid in the region outside the discontinuous surface, so-called Kirchhoff’s surface, but the generalized convective wave equation is valid for the whole region. For simplification, we assumed that the medium moves in +x direction with constant speed $U_0$. Then the original convective wave equation can be written as

$$\frac{1}{a_0^2} \frac{\partial^2 \phi}{\partial t^2} + 2 \frac{U_0}{a_0^2} \frac{\partial^2 \phi}{\partial t \partial x_1} + \frac{U_0^2}{a_0^2} \frac{\partial^2 \phi}{\partial x_1^2} - \nabla^2 \phi = 0 \quad (1)$$

where $\phi$ is the velocity potential, $a_0$ is the speed of sound, and $x_1$ is +x direction component of the position vector $\vec{x}=(x_1,x_2,x_3)$. Further, $\nabla_x$ means a gradient in $\vec{x}$ coordinates.

To extend the convective wave equation to the generalized convective wave equation, we represented the Kirchhoff’s surface as $f(\vec{x},t)=0$ such that $f(\vec{x},t)<0$ for the region inside the surface and $f(\vec{x},t)>0$ for the region outside the surface. Furthermore the function $f$ has set as $|\nabla_x f|=1$ to make the derivation process simple.\cite{1} By above definition, the surface normal unit vector pointing outward the surface can be written as $\vec{n}=\nabla f$.

The velocity potential is extended to inside the surface by assuming that it is zero inside the surface. We will call this extended velocity potential the generalized velocity potential to distinguish from original velocity potential. The generalized velocity potential can be written using Heaviside step function as

$$\tilde{\phi} = H(f)\phi = \begin{cases} \phi & \text{for } f > 0 \\ 0 & \text{for } f < 0 \end{cases} \quad (2)$$

Tilde over velocity potential means that it is a generalized variable. The generalized velocity potential is a discontinuous and non-differentiable variable but generalized function theory gives derivatives, so-called generalized derivatives.\cite{8} Details of generalized derivatives are in ref.[8]. By using the generalized derivatives, the generalized convective wave equation is
obtained which has sources terms which were not appeared in the original convective wave equation. The Kirchhoff’s surface is assumed arbitrarily moves. Fig. 1 describes the situation.

\[ f(\vec{x}, t) = 0 \]

observer

Figure 1 Arbitrarily moving Kirchhoff’s surface in a moving medium

Final form of generalized convective wave equation is

\[
\frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} + 2 \frac{U_0}{c_0^2} \frac{\partial^2 \phi}{\partial t \partial x_1} + \frac{U_0^2}{c_0^4} \frac{\partial^2 \phi}{\partial x_1^2} - \nabla^2 \phi = \frac{1}{a_0^2} \left( -\frac{\partial \phi}{\partial t} v_n \delta(f) - \frac{\partial}{\partial t} \left[ \phi v_n \delta(f) \right] \right)
\]

\[
+ 2 \frac{U_0}{a_0^2} \left( -\frac{\partial \phi}{\partial x_1} v_n \delta(f) + \frac{\partial}{\partial x_1} \left[ \phi n_i \delta(f) \right] \right)
\]

\[
+ \frac{U_0^2}{a_0^4} \left( \frac{\partial \phi}{\partial x_1} n_i \delta(f) + \frac{\partial}{\partial x_1} \left[ \phi n_x \delta(f) \right] \right)
\]

\[- \frac{\partial \phi}{\partial n} \delta(f) - \nabla \cdot \left[ \phi \hat{n} \delta(f) \right] \]

where \( \hat{v} \) is surface velocity, \( \delta \) is the Dirac-delta function, and \( v_n = \hat{v} \cdot \hat{n} \). If the Kirchhoff’s surface is stationary, \( \hat{v} = 0 \).

**CONVECTIVE KIRCHHOFF’S FORMULA**

A solution of the generalized convective wave equation can be obtained using the Green’s function. The Green’s function for the convective wave equation is

\[
G = \frac{\delta \left( \tau - t + R/a_0 \right)}{4\pi R^2} \tag{4}
\]

where

\[
R = -M_0 \left( x_i - y_i \right) + R^* \tag{5}
\]

\[
R^* = \left( \left( x_i - y_i \right)^2 + \beta^2 \left[ \left( x_2 - y_2 \right)^2 + \left( x_3 - y_3 \right)^2 \right] \right)^{\frac{1}{2}} \tag{6}
\]

\[ \beta = \sqrt{1 - M_0^2} \tag{7} \]

\( \tau \) is source time and \( t \) is observer time. \( \vec{y} = (y_1, y_2, y_3) \) is the source position. And, \( M_0 = U_0/a_0 \). The solution of Eq. (3) can be written as follows using Eq. (4).

\[
\phi(\vec{x}, t)
\]

\[
= -\frac{1}{a_0^2} \int_{-\infty}^{t} \int_{V} \frac{\partial \phi}{\partial t} v_n \delta(f) \frac{\delta(g)}{4\pi R^2} dV d\tau
\]

\[ -\frac{1}{a_0^2} \int_{-\infty}^{t} \int_{V} \phi v_n \delta(f) \frac{\delta(g)}{4\pi R^2} dV d\tau
\]

\[ -\frac{1}{a_0^2} \int_{-\infty}^{t} \int_{V} \phi n_i \delta(f) \frac{\delta(g)}{4\pi R^2} dV d\tau
\]

\[ + \frac{U_0}{a_0^2} \int_{-\infty}^{t} \int_{V} \frac{\partial \phi}{\partial x_1} n_i \delta(f) \frac{\delta(g)}{4\pi R^2} dV d\tau
\]

\[ + \frac{U_0^2}{a_0^4} \int_{-\infty}^{t} \int_{V} \frac{\partial \phi}{\partial x_1} n_x \delta(f) \frac{\delta(g)}{4\pi R^2} dV d\tau
\]

\[ - \int_{-\infty}^{t} \int_{V} \phi n_i \delta(f) \frac{\delta(g)}{4\pi R^2} dV d\tau
\]

\[ - \int_{-\infty}^{t} \int_{V} \phi n_x \delta(f) \frac{\delta(g)}{4\pi R^2} dV d\tau \tag{8} \]

where \( g = \tau - t + R/a_0 \).

Some integrals of Eq. (8) have temporal and spatial derivatives outside the integral. This causes inconvenience in numerical implementation and it is necessary to move to the inside of integral. The spatial derivative can be changed to a temporal derivative. Because the spatial derivative can act on \( \delta(g)/R^2 \) only, below relation can be used.
\[
\frac{\partial}{\partial x_i} \left( \frac{\delta (g)}{R^2} \right) = -\frac{1}{a_0} \frac{\partial}{\partial t} \left( \frac{\partial R \delta (g)}{\partial x_i} R^2 \right) - \frac{\partial R^* \delta (g)}{\partial x_i} R^2
\]  

(9)

The temporal derivative can be moved into the integral using the relation between observer time and retarded time. Below relation is already derived by Najafi-Yazid et al.[7]

\[
\frac{\partial}{\partial t} = \left[ \frac{1}{1 - M_R} \frac{\partial}{\partial \tau} \right]_{ret}
\]  

(10)

\[
M_R = \frac{v_i}{a_0} \frac{\partial R}{\partial y_i}
\]  

(11)

The Dirac-delta function of integral restrict the integration range to zero of the function variable. Thus, the integration for space reduces to the surface, \( f = 0 \), and the integration for times reduces to \( \tau \), which is the root of \( g = 0 \), so-called retarded time. But the integration variable is \( \tau \) and it is necessary to change the variable to \( g \). The relation can be obtained easily from

\[
g = \tau - t + \frac{R}{a_0} \tag{12}
\]

So the integrand has to be multiplied by \( 1/(1 - M_R) \).

The final form of Eq. (8) can be obtained using property of the Dirac-delta function and relations above as follows.

\[
4\pi \phi = \int_S \left[ A \frac{\partial \phi}{\partial \tau} + B \phi \right]_{ret} dS + \int_S \left[ C \frac{\partial \phi}{\partial n} + D \frac{\partial \phi}{\partial x_i} n_i \right]_{ret} dS
\]  

(13)

where

\[
A = -\frac{v_i}{a_0} - \frac{a_0 R^* (1 - M_R)}{a_0 R^* (1 - M_R)^2} + \frac{M_R^2 n_i \frac{\partial R}{\partial x_i}}{a_0 R^* (1 - M_R)^2} + n_i \frac{\partial R}{\partial x_i}
\]  

(14)

\[
B = B_1 + B_2 + B_3 + B_4\]

(15)

\[
B_1 = \frac{a_0 R^* (1 - M_R)}{a_0 R^* (1 - M_R)^2} \frac{\partial R}{\partial x_i} n_i
\]

\[
-\frac{M_R^2 n_i \frac{\partial R}{\partial x_i}}{a_0 R^* (1 - M_R)^2} + n_i \frac{\partial R}{\partial x_i}
\]

(16)

\[
B_2 = \frac{v_i}{a_0} - 2M_R n_i + \frac{M_R^2 n_i \frac{\partial R}{\partial x_i}}{a_0 R^* (1 - M_R)^2} + n_i \frac{\partial R}{\partial x_i} M_R
\]

(17)

\[
B_3 = -\frac{M_R^2 n_i \frac{\partial R}{\partial x_i}}{a_0 R^* (1 - M_R)^2} + n_i \frac{\partial R}{\partial x_i} M_R
\]

(18)

\[
B_4 = -\frac{M_R^2 n_i \frac{\partial R}{\partial x_i}}{a_0 R^* (1 - M_R)^2} - \frac{M_R^2 n_i \frac{\partial R}{\partial x_i}}{a_0 R^* (1 - M_R)^2}
\]

(19)

\[
C = -\frac{1}{R^* (1 - M_R)}
\]

(20)

\[
D = -\frac{2M_R v_i}{a_0} + M_R^2 n_i
\]

(21)

where dot over variables means temporal derivative.

The final form is arranged to the same form of the classical Kirchhoff’s formula except \( \partial \phi / \partial x_i \) term.

**NUMERICAL IMPLEMENTATION**

Due to the moving medium effect, an additional term, \( \partial \phi / \partial x_i \), is appeared. So, more information is required to use the convective Kirchhoff’s formula and it causes inconvenient. So, we adopted Wu et al.[3] and Guo’s[9] technique to remove this additional term. Wu et al. and Guo decomposed the additional term to the surface normal and tangential components and \( \partial \phi / \partial x_i \) can be changed to a combination of \( \partial \phi / \partial n \) and \( \phi \). \( \partial \phi / \partial x_i \) can be rewritten in vector form as follows.
\[ \int_S \left[ \frac{-2M_n \tilde{v}_{\alpha} + M_n^2 \tilde{n}_1 \partial \phi}{R' (1 - M_R)} \right]_{ret} dS \]

\[ \int_S \left[ \frac{-2 \tilde{v}_{\alpha} + M_n \tilde{M}_0 \cdot \nabla_s \phi}{R' (1 - M_R)} \right]_{ret} dS \]

where \( M_n = \tilde{M}_0 \cdot \tilde{n} \). And decomposed form is

\[ \int_S \left[ \frac{-2 \tilde{v}_{\alpha} + M_n \tilde{M}_0 \cdot \nabla_s \phi}{R' (1 - M_R)} \right]_{ret} dS \]

\[ = \int_S \left[ \frac{-2 \tilde{v}_{\alpha} + M_n \tilde{M}_0 \cdot \nabla_s \phi}{R' (1 - M_R)} \right]_{ret} dS \]

\[ = \int_S \left[ \frac{-2 \tilde{v}_{\alpha} + M_n \tilde{M}_0 \cdot \nabla_s \phi}{R' (1 - M_R)} \right]_{ret} dS \]

where \( \tilde{M}_s = \tilde{M} - \tilde{M}_0 \cdot \tilde{n} \), \( \nabla_s = \nabla - \partial / \partial n \).

More manipulation can be applied to the last integral of Eq. (23) to place the velocity potential outside the gradient.

\[ \int_S \left[ \frac{-2 \tilde{v}_{\alpha} + M_n \tilde{M}_0 \cdot \nabla_s \phi}{R' (1 - M_R)} \right]_{ret} dS \]

\[ = \int_S \left[ \frac{-2 \tilde{v}_{\alpha} + M_n \tilde{M}_0 \cdot \nabla_s \phi}{R' (1 - M_R)} \right]_{ret} dS \]

\[ \text{VERIFICATION} \]

For the verification of the formula, several cases are tested. The first case is stationary monopole in a moving medium.

The velocity potential of monopole in a moving medium is

\[ \phi(x, t) = \frac{A}{4\pi R} \exp \left[ \frac{i\omega}{t - R/a_0} \right] \]  

The amplitude of the monopole is \( A = 1 \text{m}^2 \text{s}^{-1} \) and the frequency is 10 Hz. The Mach number of moving medium is 0.5. The Kirchhoff’s surface enclose the monopole and the velocity potential and its derivatives at the Kirchhoff’s surface is obtained analytically from Eq. (30). A geometrical distance between the monopole and the observer is 50m. Fig. 2 shows directivity pattern of the RMS velocity potential. The solid line indicates the analytic solution and the symbol indicates the numerical result. The numerical result shows good agreement with the analytic solution. Although the RMS value shows good agreement with analytic solution, phase can be different. Fig. 3 shows the time
history at $\theta = 90^\circ$. The time history shows that the numerical result matches well with analytic solution not only the amplitude but also the phase.

![Figure 2 Directivity pattern of a point monopole in a moving medium](image2.png)

Figure 2 Directivity pattern of a point monopole in a moving medium

![Figure 3 Time history of a point monopole at $\theta = 90^\circ$](image3.png)

Figure 3 Time history of a point monopole at $\theta = 90^\circ$

The next case is a rotating monopole in a moving medium. The monopole strength, the frequency, and the Mach number are same as for the stationary monopole case. The monopole rotate xy-plane and the radius is 1m.

**BOUNDARY INTEGRAL EQUATION**

The Kirchhoff’s formula can be used as the boundary integral equation by set the observer position on the surface. In that case, LHS of Eq. (25) will be $2\pi \theta^{10}$ and it can be solved numerically using linear system solver. A scattering problem can be solved using the boundary integral equation.

The boundary integral equation can be used for aerodynamic problem also. The convective wave equation can be regarded as the unsteady small perturbation potential flow equation. If the body is thin, a small perturbation assumption is valid. Thus the equation can be effective to analyze the aerodynamics and aeroacoustics problems of thin propeller blades and rotor blades in cruise flight condition. By using this equation, aerodynamics and aeroacoustics can be solved simultaneously.

**CONCLUSION**

Convective Kirchhoff’s formula for arbitrarily moving surface has been derived in a physical coordinates using the generalized function theory. The derived formula do not require any coordinate transformation and it is convenient to implement numerically. The formula has additional term compared to the classical Kirchhoff’s formula due to a moving medium effect. This additional term has been manipulated by decomposition to the surface normal and tangential components. And the final formula has been arranged to the same form of the classical Kirchhoff’s formula.

The formula was verified by several test cases. Numerical results using convective Kirchhoff’s formula matched well with the analytic solution.

The formula can be used as the boundary integral equation. A scattering problem can be solved using the boundary integral equation. Furthermore the equation can be used for aerodynamic analysis for thin body. In this case, the aerodynamics and aeroacoustics of the body can be solved simultaneously.
REFERENCES


