Corotational Formulation for Nonlinear Static and Dynamic Analysis of Thin Shells with Finite Rotation

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ABSTRACT

Within the framework of corotational formulation, a three-node triangular linear flat thin shell element was extended to the geometric nonlinear static and dynamic analysis of thin shells with large translation and finite rotation. A set of energy-conserving/decaying time integration schemes has been established for nonlinear static and dynamic response analysis of thin shells involving finite rotation. The analysis of several classical numerical examples has demonstrated that the established formulations in this paper can accurately solve the geometrical nonlinear static and dynamic responses of thin shells.

INTRODUCTION

Thin shell structures with their superior mechanical performance are widely used in structural engineering, because of their loading often accompanied with large displacement and finite rotation geometric non-linear phenomena, to develop an efficient and practical non-linear finite element analysis method is still an important research issue.

The corotational formulation (CR), viewed as an alternative way of deriving efficient nonlinear finite elements for geometric nonlinear analysis with finite rotation and small strain, was initially introduced by Wempner [1] and Belytschko et al. [2], and has generated an increased amount of interest in the last two decades. The main ideal of the CR formulation is to decompose the motion of an element into two steps [3]: firstly, a rigid body rotation and translation, and then a pure deformational response. This is accomplished by using a local reference frame that continuously rotates and translates with the element. The deformational response was captured at the level of the local reference frame. Assuming the pure deformational part to be small, a geometrical linear finite element formulation can be adopted in the local frame system. Consequently, the geometric nonlinearity induced by the large rigid body motion, was completely incorporated in a corotational transformation matrix relating local and global tangent stiffness matrices and internal force vectors.

Until now, several representative works proposing the CR formulation for nonlinear static analysis of shells are given by Nour-Omid and Rankin [4], Crisfield [5], Pacoste [6], Felippa and Haugen [7] and Battini [8]. With regard to the CR formulation for nonlinear dynamic analysis of shells, only few studies have been conducted, such as Zhong and Crisfield [9], Meek and Wang [10] and Almeida and Awruch [11]. For nonlinear dynamic analysis in the framework of CR formulation, there are two basic issues need to be considered carefully, an appropriate spatial interpolation for the inertia part and a
stable time integration scheme. In the existing studies\cite{9-11}, separate interpolations for the inertia part and the elastic part were adopted. The inertia part was interpolated in the fixed global coordinate system with a linear form of global quantities, resulted in a constant mass matrix. While the elastic part was linearly interpolated in the local coordinate system by virtue of the CR formulation. As concerns the time integration schemes, an energy-conserving type scheme, called as approximately energy conservation scheme was developed for the CR formulation. This scheme was designed to approximately preserve the total energy on the level of algorithm, which guarantees the numerical stability in nonlinear dynamic analysis.

In our earlier work\cite{12}, a 3 nodes 18 degrees of freedom triangular linear flat shell element was extended to geometric nonlinear static analysis in the context of CR formulation. Recently, within the framework of the Generalized-\(\alpha\) method, by introducing an additional numerical dissipation term, an energy conserving algorithm and an energy decaying algorithm were developed for the corotational finite element nonlinear dynamic responses analysis of thin shells\cite{13}. In the present paper, the static and dynamic CR finite element formulation is briefly described firstly, and then several numerical examples are examined to demonstrate the performance of the presented formulation.

COROTATIONAL FORMULATION

Shell element kinematics

In order to describe the element motion in the CR formulation, three element configurations and two types of reference frames are defined as follows. Two actual configurations are the base configuration \(C^0\) and the deformed configuration \(C^D\), an additional intermediate configuration named corotated configuration \((C^R)\) is defined between \(C^0\) and \(C^D\). The corotated configuration is obtained from \(C^0\) through a rigid body motion. The displacements and deformations are referred to two types of reference frames: the global frame \(G\) and the local frame.

The local frame in the base and corotated configuration are denoted by \(E_0\) and \(E\), respectively.

\[\text{Fig.1 Element kinematics and coordinate systems}\]

Figure 1 shows the motion of a triangular shell element from the initial to the deformed configuration; i.e. \(C^0 \rightarrow C^D\). In the CR formulation, this process is split into two steps. First, a rigid body motion from the initial to the corotated configuration. The rigid translation is defined by the displacement of \(C\), denoted by \(u_C\). The rigid rotation is defined by an orthogonal matrix \(R_C\), using compound rotation formula, which is given as \(R_C = T_C^T T_C\).

Second, a deformational displacement in the local frame, the local deformational translation and rotation was denoted by \(u_{da}\) and \(\theta_{da}\) respectively.

The local deformational displacement and rotation are calculated as follow

\[\bar{u}_{da} = T(u_C + x^0_C - u_C - x^0_C) - \bar{x}^0_{C^0}\] (1)

\[\bar{R}_{da} = TR_{da} T_0^T\] (2)

The rotation vector \(\bar{\theta}_{da}\) can be extracted.
from the rotation matrix \( \mathbf{R}_{da} \), and then the local deformational displacement vector at node \( a \) can be expressed as

\[
\mathbf{p}_{da} = (\mathbf{u}_{da}, \mathbf{d}_{da})^T \tag{3}
\]

Further, for an element that is collected in the vector \( \mathbf{p}_d = (\mathbf{p}_{d1}, \mathbf{p}_{d2}, \mathbf{p}_{d3})^T \).

In the global frame, the element tangent stiffness is defined as the variation of the internal forces \( f \) with respect to element global degrees of freedom

\[
\delta f = K_T \delta d, \quad K_T = \frac{\partial f}{\partial d} \tag{4}
\]

Therefore, to obtain \( K_T \), it is necessary to establish the connection between \( \delta d \) and \( \delta \mathbf{p}_d \), where \( \delta d = \{\delta d_1, \delta d_2, \delta d_3\}^T \),

\[
\delta d = \{\delta u_a^T, \delta \omega_a^T\}^T .
\]

Taking the variations of \( \mathbf{u}_{da} \) and \( \mathbf{R}_{da} \), and after some tedious algebraic manipulations, a corotational transformation matrix was derived by Felippa and Haugen\(^{(7)}\). With this matrix, the relationship between \( \delta d \) and \( \delta \mathbf{p}_d \) is established as follow

\[
\delta \mathbf{p}_d = \Lambda \delta d, \quad \Lambda = \mathbf{H} \mathbf{P}_e \tag{5}
\]

where \( \mathbf{H} \) and \( \mathbf{P} \) is the tangent transformation operator and projection operator respectively, for details about those matrices see Felippa and Haugen\(^{(7)}\).

**Internal forces and tangent stiffness**

In the element corotated frame, the local deformational displacements \( \mathbf{p}_d \) are connected with the internal forces \( f \) by the element linear stiffness matrix \( \mathbf{K} : \)

\[
\mathbf{f} = \mathbf{K} \mathbf{p}_d \tag{6}
\]

where

\[
\mathbf{p}_d = (\mathbf{u}_d^T, \mathbf{d}_d^T)^T
\]

\[
\mathbf{f} = (\mathbf{f}_d^T, \mathbf{m}_d^T)^T
\]

where \( \mathbf{u}_d \) and \( \mathbf{m}_d \) are the nodal forces and moments, respectively. For the finite element formulation in the local frame, the optimal (OPT) membrane element\(^{(14)}\) with drilling degree of freedom and discrete Kirchhoff triangle (DKT) bending plate element\(^{(15)}\) are adopted and combined, obtaining a three-node triangular linear flat thin shell element.

The nodal internal forces of an element in the local and global frame can be connected by equating virtual work, i.e., \( \delta \mathbf{f} = \delta \mathbf{p}_d \mathbf{F} \). Taking into account Eq. (5), this leads to

\[
\mathbf{f} = \Lambda^T \mathbf{F} = \mathbf{T}_e^T \mathbf{P}^T \mathbf{H}^T \mathbf{F} \tag{8}
\]

According to the definition of the consistent tangent stiffness matrix in Eq. (4), the variation of the internal forces is needed

\[
\delta f = \delta \Lambda^T \mathbf{F} + \Lambda^T \delta \mathbf{F} \tag{9}
\]

For the first term in right-hand side (RHS) of Eq. (9), substituting from Eq. (6) gives\(^{(7,11)}\)

\[
\Lambda^T \delta \mathbf{F} = \Lambda^T \mathbf{K} \mathbf{A} \delta d = \mathbf{K}_M \delta d \tag{10}
\]

where \( \mathbf{K}_M \) is the material stiffness matrix in the global frame, it is a component of the tangent stiffness matrix.

For the second term in RHS of Eq. (9), taking the variation of the corotational transformation matrix defined as Eq. (5), and then we can obtain the geometric stiffness matrix \( \mathbf{K}_G \) as follows

\[
\delta \Lambda^T \mathbf{F} = (\delta \mathbf{T}_e^T \mathbf{F}^T \mathbf{H}^T + \mathbf{T}_e^T \delta \mathbf{F}^T \mathbf{H}^T + \mathbf{T}_e^T \mathbf{P}^T \delta \mathbf{H}^T) \mathbf{F} \tag{11}
\]

\[
= (\mathbf{K}_{GR} + \mathbf{K}_{GB} + \mathbf{K}_{GM}) \delta d = \mathbf{K}_{GR} \delta d
\]

The rotational geometric stiffness\(^{(7)}\) \( \mathbf{K}_{GR} \) caused by the rigid body rotation of the element, which induces the variation of the internal forces in the global frame, It is given by

\[
\mathbf{K}_{GR} = -\mathbf{T}_e^T \mathbf{F}_{nm} \mathbf{G} \mathbf{T}_e \tag{12}
\]

where

\[
\mathbf{G} = [\mathbf{G}_{u,1}, 0, \mathbf{G}_{u,2}, 0, \mathbf{G}_{u,3}, 0]
\]

\[
\mathbf{F}_{nm} = [-\mathbf{m}_{P,1}, -\mathbf{m}_{P,2}, \cdots, -\mathbf{m}_{P,3}, -\mathbf{m}_{P,3}]^T
\]

where \( \mathbf{m}_{P,a} \) and \( \mathbf{m}_{P,a} \) are the projected internal forces and moments at node \( a \),
respectively, computed from \( \mathbf{f}_e = \mathbf{P}_e^T \mathbf{H}_e^T \mathbf{f} \).

The equilibrium projection geometric stiffness \( K_{GP} \) arises from the variation of the projector with respect to the deformed element geometry, it is written as

\[
K_{GP} = -T_e^\gamma \mathbf{P}_e^T \mathbf{P}_e T_e \quad (14)
\]

where

\[
\mathbf{P}_e = [-\mathbf{n}_{p,1}, 0_3, -\mathbf{n}_{p,2}, 0_3, -\mathbf{n}_{p,3}, 0_3] \quad (15)
\]

The moment correction geometric stiffness \( K_{GM} \) due to the variation of the matrix \( \mathbf{H} \), the expression is

\[
K_{GM} = T_e^\gamma \mathbf{P}_e^T \mathbf{M} \mathbf{P}_e T_e \quad (16)
\]

where

\[
\mathbf{M} = \text{diag}[0_3, \mathbf{M}_1, 0_3, \mathbf{M}_2, 0_3, \mathbf{M}_3] \quad (17)
\]

The sub-block matrix at node \( a \), \( \mathbf{M}_a \), is formulated as

\[
\mathbf{M}_a = \eta \left[ \begin{pmatrix} 0_e \mathbf{m}_a \end{pmatrix} \begin{pmatrix} I_3 + \bar{\mathbf{d}}_a \mathbf{m}_a^T - 2 \mathbf{m}_a \overline{\mathbf{d}}_a^T \end{pmatrix} \right] + \mu \begin{pmatrix} \bar{\mathbf{d}}_a \end{pmatrix} \begin{pmatrix} \mathbf{m}_a \overline{\mathbf{d}}_a - \frac{1}{2} \mathbf{m}_a \end{pmatrix} \mathbf{H} \begin{pmatrix} \overline{\mathbf{d}}_a \end{pmatrix} \quad (18)
\]

where the fractions \( \eta, \mu \) are computed as functions of \( \overline{\mathbf{d}}_a \) \(^7\).

According to previous equations, the complete form of the element tangent stiffness and internal force vectors are determined as

\[
K_e = K_M + K_{GR} + K_{GP} + K_{GM}
\]

\[
f = T_e^\gamma \mathbf{P}_e^T \mathbf{H}_e^T \mathbf{f}
\]

**NONLINEAR DYNAMIC EQUILIBRIUM EQUATIONS**

At time \( t_{n+1} \), the nonlinear dynamic equilibrium equations for a three-node flat shell element is expressed as

\[
\mathbf{g}_{e,n+1} = \mathbf{f}_{max,n+1} + \mathbf{f}_{int,n+1} - \mathbf{f}_{ext,n+1} = 0 \quad (20)
\]

where \( \mathbf{g}_{e,n+1} \) are the equivalent dynamic out-of-balance forces of the element, \( \mathbf{f}_{max,n+1} \) the inertia forces, \( \mathbf{f}_{int,n+1} \) the internal forces and \( \mathbf{f}_{ext,n+1} \) the external applied loads.

Except for the external applied loads, the other forces in Eq. (21) can be computed as follows\(^{[7,11,13]}\)

\[
\mathbf{f}_{max,n+1} = \mathbf{R}_{e,n+1} \mathbf{M} \ddot{\mathbf{d}}_{n+1} \quad (21)
\]

\[
\mathbf{f}_{int,n+1} = \mathbf{A}_{n+1}^{T} \mathbf{f}_{int,n+1} \quad (22)
\]

where

\[
\mathbf{R}_{e,n+1} = \text{diag}[\mathbf{I}_1, \mathbf{R}_{1,1,n+1}, \mathbf{I}_1, \mathbf{R}_{2,1,n+1}, \mathbf{I}_1, \mathbf{R}_{3,1,n+1}] \quad (23)
\]

\[
\ddot{\mathbf{d}}_{n+1} = \begin{pmatrix} \ddot{\mathbf{d}}_{1,n+1}^T, \ddot{\mathbf{d}}_{2,n+1}^T, \ddot{\mathbf{d}}_{3,n+1}^T \end{pmatrix}^T \quad (24)
\]

\[
\mathbf{d}_{a,n+1} = \begin{pmatrix} \mathbf{a}_{a,n+1}^{T}, \mathbf{A}_{a,n+1}^{T} \end{pmatrix}^{T}
\]

where \( \mathbf{M} \) is the element mass matrix, \( \ddot{\mathbf{d}}_{n+1} \) is the acceleration vector of the element, \( \mathbf{a}_{a,n+1} \) is the translation acceleration and velocity vector of node \( a \), \( \mathbf{A}_{a,n+1} \) is the angular acceleration vector of node \( a \) in a body-attached frame.

As mentioned earlier, the linear global quantities interpolation was employed for the inertia part. For a triangular flat shell element, the resulted constant mass matrix is given by\(^{[9,11,13]}\)

\[
\mathbf{M} = \begin{bmatrix}
2\mathbf{M}_v & 0_3 & \mathbf{M}_v & 0_3 & \mathbf{M}_v & 0_3 \\
0_3 & 2\mathbf{M}_w & 0_3 & \mathbf{M}_w & 0_3 & \mathbf{M}_w \\
\mathbf{M}_v & 0_3 & 2\mathbf{M}_v & 0_3 & \mathbf{M}_v & 0_3 \\
0_3 & \mathbf{M}_w & 0_3 & 2\mathbf{M}_w & 0_3 & \mathbf{M}_w \\
\mathbf{M}_v & 0_3 & \mathbf{M}_v & 0_3 & 2\mathbf{M}_v & 0_3 \\
0_3 & \mathbf{M}_w & 0_3 & \mathbf{M}_w & 0_3 & 2\mathbf{M}_w 
\end{bmatrix}
\]

where \( \mathbf{0}_3 \) is the \( 3 \times 3 \) zero matrix. \( \mathbf{M}_v \) and \( \mathbf{M}_w \) are given by

\[
\mathbf{M}_v = \frac{\rho A h}{12} \mathbf{I}_3 \quad \mathbf{M}_w = \frac{\rho A h}{6} \mathbf{I}_3
\]

where \( \rho \), \( A \) and \( h \) represent mass density, element area and thickness, respectively. \( \mathbf{I}_3 \) is the \( 3 \times 3 \) identity matrix.

At the end of the time interval \( \Delta t \) from time \( t_n \) until \( t_{n+1} \), the velocity and acceleration vectors for each element are updated by using the standard Newmark formulae:

\[
\begin{align}
\dot{\mathbf{d}}_{n+1} &= \frac{1}{\rho \mu} \mathbf{R}_{e,n}^{T} \mathbf{A} \Delta \mathbf{p} - \left( \frac{1}{2 \beta} - 1 \right) \dot{\mathbf{d}}_n - \frac{1}{2} \ddot{\mathbf{d}}_n \\
\ddot{\mathbf{d}}_{n+1} &= \frac{1}{\rho \mu} \mathbf{R}_{e,n}^{T} \mathbf{A} \Delta \mathbf{p} - \frac{1}{\rho \mu} \dot{\mathbf{d}}_n - \left( \frac{1}{2 \beta} - 1 \right) \ddot{\mathbf{d}}_n
\end{align}
\]

where \( \mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2 \beta} \end{pmatrix} \) and \( \Delta t \) is the time step.
Where
\[ \Delta p = \{\Delta p_1^T, \Delta p_2^T, \Delta p_3^T\}^T \] (28)
\[ \Delta p_a = \{\Delta u_1^T, \Delta \dot{u}_1^T\}^T \]

In the following derivations, the variations of nodal velocity and acceleration are needed. Taking the variation of Eq. (28) gives
\[ \dot{\Delta d}_{n+1} = \frac{1}{\beta \Delta t^2} R_{e,n}^T \mathbf{H}_{\Delta \theta} \Delta \dot{d}_{n+1} \] (29)
\[ \ddot{\Delta d}_{n+1} = \frac{1}{\beta \Delta t^2} R_{e,n}^T \mathbf{H}_{\Delta \theta} \Delta \ddot{d}_{n+1} \]

ENERGY CONSERVING AND DECAYING ALGORITHMS

With the application of generalized-\( \alpha \) method, the nonlinear dynamic equilibrium equations, Eq. (21) can be rewritten as
\[ \mathbf{f}_{e,n+1} = \mathbf{f}_{\text{int},n+1} - \mathbf{f}_{\text{ext},n+1} = 0 \] (30)
where
\[ \mathbf{f}_{\text{int},n+1} = (1 - \alpha_m) \mathbf{f}_{\text{int},n} + \mathbf{f}_{\text{ext},n} \] (31)
\[ \mathbf{f}_{\text{ext},n+1} = (1 - \alpha_f) \mathbf{f}_{\text{ext},n} + \mathbf{f}_{\text{int},n} \] (32)
where \( \Lambda_{n+1-a_f} \) is defined as the corotational operator matrix at a generalized point \( n+1-\alpha \), it is given by
\[ \Lambda_{n+1-a_f} = (1 - \alpha_f) \Lambda_{n+1} + \alpha_f \Lambda_n \] (34)
Since \( \mathbf{p}_{d,n+1-a_f} \) is a small deformation, which leads to
\[ \mathbf{p}_{d,n+1-a_f} = (1 - \alpha_f) \mathbf{p}_{d,n+1} + \alpha_f \mathbf{p}_{d,n} \]
\[ \mathbf{f}_{\text{int},n+1} \mathbf{p}_{d,n+1-a_f} = (1 - \alpha_f) \mathbf{f}_{\text{int},n+1} + \alpha_f \mathbf{f}_{\text{int},n} \] (35)

According to the ideal of energy conservation, in a time interval integration procedure, the change of kinematic (\( \Delta K \)) and strain (\( \Delta S \)) energy of the system should be equal to the work done by the external applied loads (\( \Delta W \)). This energy conservation condition can be achieved with the dynamic equilibrium equations at the general point \( n+1-\alpha \) as follows
\[ \Delta E = \Delta K + \Delta S - \Delta W \]
\[ = \{f_{\text{int},n+1} - f_{\text{int},n} - f_{\text{ext},n+1} + f_{\text{ext},n}\} \Delta d \] (36)
\[ = \mathbf{g}_{e,n+1-a} \Delta d = 0 \]

In the above equation, if \( \mathbf{g}_{e,n+1-a} = 0 \), then \( \Delta E = 0 \), which means the total energy of the system is conservative.

The four parameters \( \alpha_m, \alpha_f, \beta, \gamma \), can be reduced into a single parameter, the spectral radius \( \rho_\infty \), by using the following equations
\[ \alpha_m = \frac{2 \rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1} \]
\[ \beta = \frac{1}{2}(1 - \alpha_m + \alpha_f)^2, \quad \gamma = \frac{1}{2} - \alpha_m + \alpha_f \] (37)
Since the numerical dissipation introduced with the choice \( \rho_\infty \neq 1 \) does not always extend to the nonlinear regime, an additional numerical dissipation given by Armero and Petocz\[16\] is introduced into the presented algorithms. The additional numerical dissipation coefficient \( \xi \) only influences the calculation of the internal forces in the local coordinate system
\[ \mathbf{f}_{\text{int},n+1-a_f} = \Lambda_{n+1-a_f} \mathbf{f}_{\text{int},n+1} + (\alpha_f - \xi) \mathbf{f}_{\text{int},n+1-a_f} \] (38)
Consequently, the time integration algorithms expressed within Eqs. (30) and (38), are the energy conserving and decaying algorithms. The choice \( \rho_\infty = 1, \xi = 0 \), yields the energy conserving algorithm, which is identical to the approximately energy conservation algorithm\[9,11\]. The choice \( \rho_\infty < 1, \xi > 0 \), leads to the energy decaying algorithm.

If the Newton-Raphson iteration procedure is used to solve the nonlinear dynamic equilibrium equations, we need to perform the consistent linearization to provide the exact tangent operator. At the \( (i+1)^{th} \) iteration of the time instant \( t_{n+1} \), taking the first-order truncated Taylor series for Eq. (30) at the solution of the \( (i)^{th} \) iteration,
and the variations of $\mathbf{f}_{\text{max}, n+1-\alpha_{u}}$ and $\mathbf{f}_{\text{int}, n+1-\alpha_{f}}$, respectively. Replacing $\mathbf{f}_{\text{int}, n+1-\alpha_{f}}$ at $t_{n+1}$ will be solved in the following steps.

Taking the variation of Eq. (31) gives

$$\delta \mathbf{f}_{\text{max}, n+1-\alpha_{u}} = (1 - \alpha_{u}) \delta \mathbf{R}_{e,n+1} \ddot{\mathbf{M}}_{n+1} + (1 - \alpha_{u}) \mathbf{R}_{e,n+1} \mathbf{M} \delta \ddot{\mathbf{d}}_{n+1} \quad (41)$$

$$= \mathbf{K}_{\text{max}, n+1-\alpha_{u}} \delta \mathbf{d}_{n+1}$$

For the first term of the RHS of Eq. (41), using the relation $\delta \mathbf{R}_{n,n+1} = \ddot{\mathbf{M}}_{n+1}$, we can rewritten this term as

$$(1 - \alpha_{u}) \delta \mathbf{R}_{e,n+1} \ddot{\mathbf{M}}_{n+1} = \text{diag} \{ 0_{1}, \mathbf{m}_{\text{max},1,n+1}^{*}, \ldots, 0_{3}, \mathbf{m}_{\text{max},3,n+1}^{*}\} \delta \mathbf{d}_{n+1} \quad (42)$$

Where

$$\mathbf{f}'_{\text{max}, n+1} = (1 - \alpha_{u}) \mathbf{R}_{e,n+1} \mathbf{M} \ddot{\mathbf{d}}_{n+1} \quad (43)$$

For the second term of the RHS of Eq. (41), the substitution of $\delta \ddot{\mathbf{d}}_{n+1}$ in equation (29) leads to

$$(1 - \alpha_{u}) \mathbf{R}_{e,n+1} \mathbf{M} \delta \ddot{\mathbf{d}}_{n+1} = -\frac{1 - \alpha_{u}}{\rho \mathbf{u}_{n+1}} \mathbf{R}_{e,n+1} \mathbf{M} \mathbf{R}_{e,n+1} \mathbf{H}_{e,n+1} \delta \mathbf{d}_{n+1} = \mathbf{K}_{\text{int}, 2} \delta \mathbf{d}_{n+1} \quad (44)$$

Taking the variation of Eq. (38) gives

$$\delta \mathbf{f}_{\text{int}, n+1-\alpha_{f}} = (1 - \alpha_{f}) \delta \mathbf{A}_{n+1}^{T} \mathbf{f}_{\text{int}, n+1-\alpha_{f}}, \xi$$

$$+ \mathbf{A}_{n+1-\alpha_{f}}^{T} (1 - \alpha_{f} + \xi) \delta \mathbf{f}_{\text{int}, n+1} = \mathbf{K}_{\text{int}, 3} \delta \mathbf{d}_{n+1} \quad (45)$$

For the first term of the RHS of Eq. (46), according to the CR formulation [7,11,12], we can obtain

$$(1 - \alpha_{f}) \delta \mathbf{A}_{n+1}^{T} \mathbf{f}_{\text{int}, n+1-\alpha_{f}}, \xi$$

$$= (1 - \alpha_{f}) \mathbf{K}_{G}^{*} (\mathbf{f}_{\text{int}, n+1-\alpha_{f}}, \xi) \delta \mathbf{d}_{n+1} \quad (46)$$

$$= \mathbf{K}_{\text{int}, 3} \delta \mathbf{d}_{n+1}$$

It should be noted that, the way to construct the matrix $\mathbf{K}_{G}^{*} (\mathbf{f}_{\text{int}, n+1-\alpha_{f}}, \xi)$ is the same as given in Eq. (11), but using the corotational operator matrix $\mathbf{A}_{n+1}$ corresponding to the configuration at time $t_{n+1}$ and replacing $\mathbf{f}$ with $\mathbf{f}_{\text{int}, n+1-\alpha_{f}}, \xi$.

The second term of the RHS of Eq. (39) can be further written as

$$\mathbf{A}_{n+1-\alpha_{f}}^{T} (1 - \alpha_{f} + \xi) \delta \mathbf{f}_{\text{int}, n+1}$$

$$= (1 - \alpha_{f} + \xi) \mathbf{A}_{n+1}^{T} \mathbf{K}_{A} \mathbf{A}_{n+1} \delta \mathbf{d}_{n+1} \quad (47)$$

Finally, we can obtain the following expressions

$$\mathbf{K}_{\text{max}, n+1-\alpha_{u}} = \mathbf{K}_{\text{max}, 1} + \mathbf{K}_{\text{max}, 2}$$

$$\mathbf{K}_{\text{int}, n+1-\alpha_{f}} = \mathbf{K}_{\text{int}, 1} + \mathbf{K}_{\text{int}, 2} \quad (48)$$

**NUMERICAL EXAMPLES**

**Top opened hemispherical shell**

Fig.2 shows the top opened hemispherical shell. The shell is loaded by alternating radial point forces $P = 400N$ at 90° intervals. There is a 18° circular cutout at its pole. The geometric parameters are: radius $R = 10m$ and thickness $h = 0.04m$. The elastic modulus and Poisson ratio is $E = 6.825 \times 10^{7} Pa$ and $\nu = 0.3$, respectively. Due to symmetry, one-quarter of the shell is modeled, with 16×16×2 and 12×12×2 elements. From Fig.3, the following observations can be made: both meshes give close results and they are in good agreement with the results obtained by Sze et al. [17], whereas it seems that a coarse mesh gives a slightly stiffening prediction. The deformed configuration of the shell is drawn in Fig.4.
Motion of a short cylinder under impulsive line loads

This is a classical example, proposed by Simo and Tarnow[18], and used to test the ability of a presented algorithm in solving problems with large motion (displacements and rotations) for long-term computations. The initial configuration is shown in Fig. 5. The geometric parameters are: diameter $D = 15m$, height $H = 3m$ and thickness $h = 0.02m$. The material properties are: Young’s modulus $E = 210MPa$, Poisson’s ratio $\nu = 0.5$ and mass density $\rho = 1kg/m^3$. Not being constrained, the short cylinder shell subjected four impulsive line loads located in the positions described by the angles $0^\circ, 90^\circ, 180^\circ$ and $270^\circ$. These loads are: $F_0 = F_{270} = \{0,-1,-1\}^T f(t)$ (N) and $F_{90} = F_{180} = \{1,1,1\}^T f(t)$ (N). The expression of $f(t)$ is given by

$$f(t) = \begin{cases} 10t, & t \leq 0.5 \\ 10 - 10t, & 0.5 < t \leq 1 \\ 0, & t > 1 \end{cases} \text{ (49)}$$

The shell was meshed into $28 \times 3 \times 2$ finite elements. The time step was $\Delta t = 0.02s$ and the total computational time was $25s$. Under the action of the external loads, the shell exhibits large complex motion in the 3D space. Deformed configurations of the shell at various time instants were illustrated in Fig. 6. Evolutions of displacements at point A were presented in Fig. 7. The displacement curves at point A were plotted only for $t = 0 \sim 6s$. It can be observed that the presented results of the Energy Conserving algorithm are practically identical to the results given by Almedia et al.[11].

The time histories of different energies were plotted in Fig. 8. After the loads have been removed, both the EC algorithm and the HHT-\(\alpha\) scheme conserve the total energy of the shell. While the Newmark scheme...
produces an uncontrolled increase of the total energy around \( t = 8.42s \), results in a non-convergent solution. However, the results demonstrated that the presented algorithms have the ability to solve those problems involving large displacements and rotations for long-term computations.

Dynamic buckling of a cylindrical shell
This example, first presented by Kuhl and Ramm\textsuperscript{[19]}, was used to examine the ability of the presented algorithm in suppressing unwanted high-frequency responses with controllable numerical dissipations. The initial geometry is shown in Fig.9. The geometric parameters are: radius \( R = 5m \), length \( b = 5m \) and thickness \( h = 0.1m \). The material properties are: Young’s modulus \( E = 200\text{GPa} \), Poisson’s ration \( \nu = 0.25 \) and mass density \( 1.0 \times 10^4 \text{kg/m}^3 \). The two straight edges of the shell are simply supported, while the two curved edges are free. A transient concentrated load \( f(t) = [0, 0, -1]^T \) \( f(t) \) (N), is applied at the top of this shell. The expression of \( f(t) \) is given by

\[
f(t) = \begin{cases} 
2.5 \times 10^8 t, & t \leq 0.2 \\
5.0 \times 10^7, & t > 0.2 
\end{cases}
\]  

Due to symmetry, one-quarter of the shell was modeled by a \( 16 \times 16 \times 2 \) mesh. A time step \( \Delta t = 0.001s \) was employed and all the
computation run 0.8s. Three algorithms were considered: the Energy Conserving (EC) algorithm and the Energy Decaying algorithms (ED1: $\rho_\infty = 0.8, \xi = 0.0$ and ED2: $\rho_\infty = 1.0, \xi = 0.056$).

As the external load linearly increases from 0 to a constant value, the shell successively experiences three deformation processes, i.e., pre-buckling, buckling and post-buckling. The deformed configurations of the shell at various time instants were displayed in Fig. 10. The displacements of points A and B were plotted in Fig. 11. The present results compared well in magnitude with the results given by Kuhl and Ramm [19], but the prediction of the buckling limit-point has a phase shift about 0.01s. However, the present results are in good agreement with the results given by Bottasso et al. [20], not only in magnitude but also in phase.

The time history of the velocity component along z-direction of point A was shown in Fig. 12. The results demonstrated that, the Energy Decaying algorithms (ED1 and ED2) with additional numerical dissipations have almost no effects on the low-frequency responses before the post-buckling of the shell, however, remarkable effects on the high-frequency responses after the post-buckling occurred. The velocities of points A and B decrease rapidly and tend to zero. Relative to the choice $\rho_\infty < 1$, the choice $\xi > 0$ produces a more dissipative effect on inhibiting the unwanted high-frequency responses. The time histories of the total energy were plotted in Fig. 13. The results suggest that the additional numerical dissipation dissipates most of the total energy.

![Fig.10 Deformed configurations of the cylindrical shell at various time instants](image1)

![Fig.11 Displacements in the –z direction of points A and B](image2)

![Fig.12 Time history of the velocity component along z-direction of point A](image3)
CONCLUSION

In the framework of CR formulation, a 3 nodes 18 degrees of freedom triangular linear flat shell element was extended to geometric static and dynamic nonlinear analysis.

With the framework of the generalized-α method, by introducing an additional numerical dissipation term, an energy conserving algorithm and an energy decaying algorithm were developed for the corotational finite element nonlinear dynamic responses analysis of thin shells. Finally, three popular numerical examples were examined. The comparison of the results obtained between in this paper and in the literature, demonstrated that the presented formulation can accurately solve static and dynamic nonlinear problems of thin shells involving large translations and finite rotations.

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